CONJUGATE EQUATION OF HEAT CONDUCTION FOR AN ISOTROPIC LINEAR VISCOELASTIC BODY

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A conjugate equation of heat conduction is derived for a material which behaves viscoelastically under both shear and volume deformation.

In the derivation of the conjugate equation of heat conduction for a viscoelastic body in [1], the latter was assumed to behave viscoelastically under shear as well as under elastic deformation by hydrostatic compression or tension. Although little volume relaxation and creep occur in metals and many polymers within the linear range, such an approximation of the material is not always valid [2-4]. Generally, volume relaxation in a viscoelastic material may not be disregarded in calculations of the change in internal energy [5]. This applies, first of all, to problems where thermal dissipation of viscoelastic energy is a problem of major concern [6].

We will assume here that the behavior of the material under both shear and volume deformation is viscoelastic, with the deviators of the stress and the strain tensor related through a differential-operator equation (based on any whatever model consisting of elastic and viscous elements [4, 5, 7]):

$$P\{s_{ij}\} = Q\{e_{ij}\}, \tag{1}$$

and with the spherical components

$$P'\{\sigma\} = Q'\{\varepsilon - \alpha_T T\}. \tag{2}$$

In deriving the conjugate equation of heat conduction we use the fundamental energy equation for a continuous medium [1]:

$$\sigma_{ij}\dot{\varepsilon}_{ij} - q_{i,i} = \dot{U}. \tag{3}$$

Splitting the strain tensor ε_{ij} into the state-of-strain parameters $\varepsilon^y_{ij} = \varepsilon^y_{ij} + \varepsilon^y\delta_{ij}$ and the dissipative strain parameters

$$\varepsilon^{\text{в}}_{ij} = e^{\text{в}}_{ij} + \varepsilon^{\text{в}}\delta_{ij}, \tag{4}$$

we can rewrite Eq. (3) for an arbitrary mechanical model which represents the viscoelastic state of the body as

$$\sigma^m_{ij}\,[(\dot{\varepsilon}^y_{ij})^m + (\dot{\varepsilon}^{\text{в}}_{ij})^m] - q_{i,i} = \dot{U}. \tag{5}$$

This equation will now be replaced by an equivalent system of two equations:

$$\sigma^m_{ij}\,(\dot{\varepsilon}^y_{ij})^m + T\dot{S} = \dot{U}, \tag{6}$$

$$\sigma^m_{ij}\,(\dot{\varepsilon}^{\text{в}}_{ij})^m - q_{i,i} = T\dot{S}. \tag{7}$$

Equation (7) will be rewritten so as to take into account relation (4) as well as the resolution of the stress tensor σ_{ij} into the stress deviator s_{ij} and the hydrostatic stress component σ. Namely,

$$s^m_{ij}\,(\dot{e}^{\text{в}}_{ij})^m + \sigma^m\,(\dot{\varepsilon}^{\text{в}})^m\,\delta_{ij}\delta_{ij} + (\dot{\varepsilon}^{\text{в}})^m\,s^m_{ij}\delta_{ij} + \sigma^m\,(\dot{e}^{\text{в}}_{ij})^m\,\delta_{ij} - q_{i,i} = T\dot{S}. \tag{8}$$

Since for the viscous deviator we have $s_{ij}\delta_{ij} = 0$ and $\varepsilon_{ij}\delta_{ij} = 0$, hence the third and the fourth terms on the left-hand side of Eq. (8) are equal to zero.

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Further analogously expanding the free-energy function into a power series [1] with respect to the three principal invariants of the strain tensor ε_{ij}^{y} and the temperature T_1, then determining from Eq. (6) the relation between the entropy density and the free-energy function, we obtain from (8) the general equation of heat conduction for a viscoelastic body:

$$c_{\varepsilon}\dot{T} = \lambda_{T}T_{,ii} + s_{ij}^{m}(\dot{e}_{ij}^{B})^{m} + 3\sigma^{m}(\dot{\varepsilon}^{B})^{m} - 9\alpha_{T}KT_{0}\dot{\varepsilon}^{y}. \qquad (9)$$

The sum of the last three terms on the right-hand side of Eq. (9) represents the heat-source function, whose components define respectively: 1) the rate of heat generation due to dissipation of mechanical energy during shear creep, 2) the rate of heat generation due to dissipation of mechanical energy during volume creep, and 3) the rate of heat generation due to mechanical expansion of the material.

In the case of a Hook body, the internal source in the equation of heat conduction (9) is determined only by thermoelastic dissipation of energy. The average stresses and strains are in this case related according to the linear theory of thermoelasticity: $\sigma = 3K(\varepsilon^{y} - \alpha_{T}T)$. For a viscoelastic body, however, the volume strains are a result of superposed elastic and viscous deformations so that σ and ε are related according to the rheological equation (2).

If volume creep in the material is disregarded, then the equation of heat conduction (9) becomes the well known equation in [1].

In order to estimate the amount of mechanical energy dissipated during volume creep in the material, we rewrite Eq. (9) as

$$c_{\varepsilon}\dot{T} = \lambda_{T}T_{,ii} + s_{ij}^{m}(\dot{e}_{ij}^{B})^{m}(1+N) - 9\alpha_{T}KT_{0}\dot{\varepsilon}^{y}. \qquad (10)$$

An analysis of the ratio $N = 3\sigma^{m}(\dot{\varepsilon}^{B})^{m}/s_{ij}^{m}(\dot{\varepsilon}_{ij}^{B})^{m}$ for a Maxwell body and for a generalized linear body indicates that $0 \le N \le 0.5$. In the special case of a one-dimensional state of stress ($\nu \neq 0$) in these bodies $N = 1/2 \cdot 1 - 2\nu/1 + \nu$. As can be seen, accounting for the viscoelastic behavior of the material during volume deformations has the most pronounced effect in materials with a low Poisson ratio. The heat source due to volume creep is most powerful here when $\nu = 0$ and becomes equal to half the thermal flux dissipated during shear creep in the material.

For an incompressible medium ($\nu = 0.5$) the equation of heat conduction (10) becomes

$$c_{\sigma}\dot{T} = \lambda_{T}T_{,ii} + s_{ij}^{m}(\dot{e}_{ij}^{B})^{m} - 3\alpha_{T}T_{0}\dot{\sigma}. \qquad (11)$$

The absence of the $3\sigma^{m}(\dot{\varepsilon}^{B})^{m}$ term in Eq. (11) indicates that it is valid to disregard the effect which relaxation of the volume parameters has on the internal heat generation, if the material is also assumed incompressible.

It is to be noted that, from the energy point of view, the existence of materials with $-1 < \nu < 0$ is not impossible [8]. For such hypothetical materials $N > 0.5$ and in the special case of $\nu \to -1$ the heat-source function in Eq. (9) will be determined only by dissipation of energy due to volume creep and change in dilatation.

Thus, defining the relation between average stresses and strains in terms of the linear theory of thermoelasticity for a body with rheological properties, as has been done in [1], will distort both the quantitative and the qualitative characteristics of its thermodynamic behavior.

NOTATION

s_{ij}	is the deviator of stress tensor;
e_{ij}	is the deviator of strain tensor;
P, Q, P', Q'	are the differential operators;
p_n, q_n, f_n, h_n	are the coefficients involving material properties;
σ, ε	are the average values of stress and strain;
α_T	is the linear expansivity;
T	is the thermodynamic temperature;
σ_{ij}	is the stress tensor;
$\dot{\varepsilon}_{ij}$	is the strain-rate tensor (dot denotes a time derivative);
q_i	is the thermal flux vector;
U	is the volume density of internal energy;
ε_{ij}^{y}, ε_{ij}^{B}	are the elastic-strain and dissipative-strain tensors;

e^{y}_{ij}, e^{B}_{ij}	are the deviator of elastic-strain and of dissipative-strain tensor respectively;
ε^{y}, ε^{B}	are the average value of elastic strain and of dissipative strain respectively;
δ_{ij}	is the Kronecker delta;
m	is the number of elements in a given model;
S	is the entropy density;
$T_1 = (T - T_0)/T_0$;	
T_0	is the initial temperature of body;
c_ε	is the specific heat at constant strain;
c_σ	is the specific heat at constant stress;
λ_T	is the thermal conductivity;
K	is the modulus of volume compression;
ν	is the Poisson ratio.

LITERATURE CITED

1. B. Boley and J. Wayner, Theory of Thermal Stresses [Russian translation], Izd. Mir, Moscow (1964).
2. G. A. Teters, Compound Loading and Stability of Polymer Shells [in Russian], Izd. Zinatne, Riga (1969).
3. V. E. Gul' and V. N. Kuleznev, Structure and Mechanical Properties of Polymers [in Russian], Izd. Vysshaya Shkola, Moscow (1966).
4. G. G. Hiltin, "Theory of viscoelasticity," in: Principles of Designing Plastic Parts [Russian translation], Izd. Mashinostroenie, Moscow (1970).
5. A. A. Il'yushin and B. E. Pobedra, Mathematical Principles of Thermoviscoelasticity Theory [in Russian], Izd. Nauka, Moscow (1970).
6. V. S. Postinkov, Internal Friction in Metals [in Russian], Izd. Metallurgiya, Moscow (1969).
7. G. Parkus, Transient Thermal Stresses [Russian translation], Fizmatgiz, Moscow (1963).
8. A. I. Lur'e, Theory of Elasticity [in Russian], Izd. Nauka, Moscow (1970).